On Some Properties of the Medians of Closed Continuous Curves Formed by Analytic Arcs.

By Arnold Emch.

§ 1. *Introduction.*

In two recent articles* of this journal I have investigated the properties of the medians of closed convex curves with continuous tangents and of closed convex curves formed by a finite number of ordinary analytic arcs, including convex rectilinear polygons. By means of continuous sets of medians corresponding to a continuous set of orthogonal directions it was possible to prove the theorem that to every such curve at least one square may be inscribed in such a manner that the vertices of the square lie on the curve.

In what follows I shall generalize the result for any closed continuous curve composed of a finite number of analytic arcs with a finite number of inflexions and other singularities.

The following two propositions of analysis situs will be used in this demonstration:

I. Consider a singly connected region R of a plane, bounded by a continuous curve B formed by analytic arcs without multiple points, and assume that the points of the boundary belong to R. Any continuous curve C formed by analytic arcs without multiple points, that connects any two distinct points of the boundary and has no segment in common with the boundary, and whose points belong to R, separates this region into two distinct sub-regions R_1 and R_2. All points of R_1 and R_2, except those of C, shall belong to R.

Choose now any two points, P_1 in R_1 and P_2 in R_2, and join them by a continuous curve K, composed of analytic arcs, that has no segments or singular points in common with C, and lies entirely within R.

Such a curve K cuts C in an odd number of points. When P_1 and P_2 are both in R_1 or in R_2, then K cuts C in an even number of points.

II. Let P_1, P_2, P_3, P_4 be four distinct points of an ordinary closed continuous curve B without multiple points, which follow each other in the given order when the curve is described uniformly in the same sense, and for which

$$P_1P_2 = P_2P_3 = P_3P_4 = P_4P_1.$$

The figure formed by four such points I call an *inscribed rhomb of the first kind.*

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* Vol. XXXV, pp. 407–412; Vol. XXXVII, pp. 19–28.

Let $P'_1 P'_2 P'_3 P'_4$ be another rhomb inscribed to the same curve B, such that the vertices follow each other in the given order when the curve is again described continuously in the same sense, and

$$P'_1 P'_4 = P'_4 P'_2 = P'_2 P'_3 = P'_3 P'_1 .$$

This figure shall be called an *inscribed rhomb of the second kind.*

A continuous set of rhombs of the first kind and a continuous set of rhombs of the second kind attached to the same curve B can have no rhomb in common. In other words, it is impossible to deform a rhomb of the first kind continuously into a rhomb of the second kind.

Indeed, when $P_1 P_2 P_3 P_4$ is a rhomb of the first kind, to pass continuously in the same sense along the curve from P_1 to P_3, either the point P_2 or the point P_4 is met. Hence, a continuous change of the rhomb $P_1 P_2 P_3 P_4$ in which we pass directly along the curve from P_1 to P_3 could only occur after P_2 and P_4 had crossed P_3 into the two parts of the curve B determined by P_1 and P_2 as extremities. Hence, in the continuous deformation of the rhomb of the first kind we would have rhombs with two coincident consecutive vertices. This, however, is impossible on a curve without double points.

§ 2. *Medians or Mid-point-Curves of Systems of Parallel Chords.*

1. *Continuous Sets of Parallel Chords.* Consider first a closed continuous curve* with a continuous tangent without multiple points and with no other singularities than inflexions, and the system of all secants and tangents of this curve parallel to a given direction τ. From the definition of such a curve it follows that there are two extreme tangents between which the curve is located. In case of an oval, there are just two and no other tangents to the curve for every direction. In case of a general closed ordinary curve we shall first assume that there are no multiple or inflectional tangents for a particular given direction τ. Let A_τ and Z_τ be the points of tangency of the extreme tangents, and let the secant move parallel to τ from A_τ to Z_τ. In the neighborhood of A_τ every secant cuts the curve in two points, E_1 and E_2 (Fig. 1). In case of more than two (an even number) tangents, the secant, in moving from A_τ to Z_τ, becomes for the second time tangent to the curve at some point T_1. As this point counts for two points of intersection, there are on this secant an even number of intersections, beside this tangency. Evidently the system of parallel chords between A_τ and T_1 thus obtained forms a continuous set A_τ, $E_1 E_2$, $E'_1 E'_2$, ..., $T_1 S_1$. From this last position move the secant such that T_1 continues its motion along the curve in the same sense as from A_τ to T_1. As there is no tangency between A_τ and S_1, the T_1 must necessarily

* We restrict this expression throughout to curves as defined in the introduction.

move into a new point of tangency T_2, while the other extremity S_1 of the tangent chord moves to S_2. Again, the system of parallel chords T_1S_1, B_2S_1', ..., T_2S_2 forms a continuous set. Continuing the motion of the secant such that T_2 continues to move on the curve in the same sense as from T_1 to T_2, and if there are no other tangents between T_2S_2 and Z_τ, the chords parallel to τ between these two positions form a continuous set. In case that there are

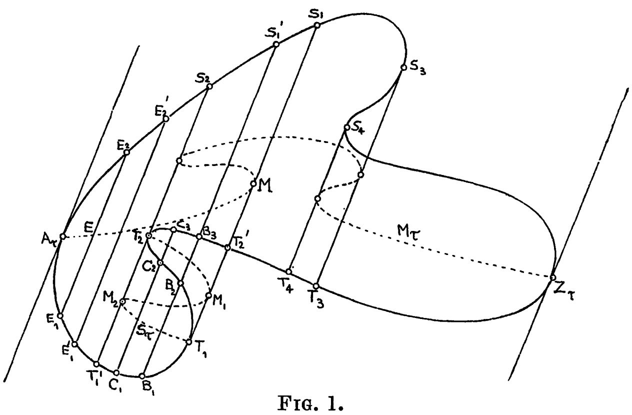

FIG. 1.

other tangents, either T_2 or S_2 will move into the next point of tangency. Suppose that S_2 move into such a point S_3, and T_2 into T_3; then the chords between T_2S_2 and T_3S_3 form also a continuous set. Continuing this process, all points of tangency, and finally Z_τ, are reached. In this manner we have formed *a continuous set of chords parallel to a given direction between the two extreme tangents.*

Since S_1T_1 was the first tangent, we see that in the piece $A_\tau S_1$ of the curve there is no tangency. Hence, there lies an S-shaped portion of the curve between the tangents at T_1 and T_2. The chords determined by this curve, $T_1'T_1T_2T_2'$, form a continuous set T_1, B_1B_2, C_1C_2, $T_1'T_2$, C_1C_3, B_1B_3

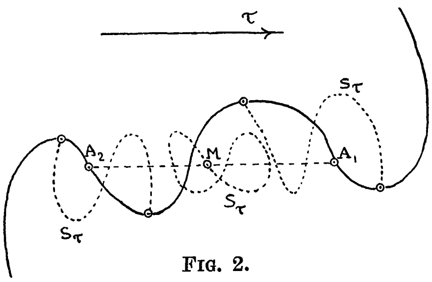

FIG. 2.

T_1T_2', B_2B_3, C_2C_3, T_2. Such a set we may call a *secondary set.* Another example of a secondary set is shown in Fig. 2, consisting of all chords forming a continuous set with the chord A_1A_2 as one of the elements.

2. *Medians.* The locus of the mid-points of the chords of the continuous set of parallel chords between A_τ and Z_τ is a continuous curve with the extremities A_τ and Z_τ. As before, this locus may be called the *median* M_τ of the

curve associated with the direction τ. The loci of mid-points of the chords of secondary sets we call *secondary medians* S_τ. Let $y=mx+b$ be the equation of a secant cutting the curve at E_1 and E_2. In the neighborhood of these points the coordinates of the points of the curve may be represented as uniform continuous differentiable functions of the parameter b:

$$x_1=f(b), \quad y_1=g(b), \quad (E_1),$$
$$x_2=h(b), \quad y_2=j(b), \quad (E_2).$$

For the coordinates of the mid-point M of the chord E_1E_2 we get

$$\xi=\tfrac{1}{2}[f(b)+h(b)],$$
$$\eta=\tfrac{1}{2}[g(b)+j(b)].$$

When b varies, these are the parametric equations of the median. Designating by x and y current coordinates, and by $(')$ derivatives with respect to b, the equations of the tangents to the curve at E_1 and E_2, and to the median at M, are

$$y_1'x-x_1'y+x_1'y_1-y_1'x_1=0,$$
$$y_2'x-x_2'y+x_2'y_2-y_2'x_2=0,$$
$$(y_1'+y_2')x-(x_1'+x_2')y+\tfrac{1}{2}(x_1'y_1+x_2'y_2-y_1'x_1-y_2'x_2+x_2'y_1+x_1'y_2-x_2y_1'+x_1y_2')=0.$$

But from $y=mx+b$ and $m=\dfrac{y_1-y_2}{x_1-x_2}$ we find by differentiation

$$y_1'=\frac{y_1-y_2}{x_1-x_2}x_1'+1, \quad y_2'=\frac{y_1-y_2}{x_1-x_2}x_2'+1,$$

and from this

$$x_1'y_1+x_2'y_2-y_1'x_1-y_2'x_2=x_2'y_1+x_1'y_2-x_2y_1'-x_1y_2'.$$

The equation of the tangent to the median assumes therefore the form

$$(y_1'+y_2')x-(x_1'+x_2')y+(x_1'y_1+x_2'y_2-y_1'x_1-y_2'x_2)=0,$$

and appears as the sum of the left-hand members of the equations of the tangents at E_1 and E_2. The three tangents are therefore concurrent.

This fact may be stated as

Theorem 1. *If E_1 and E_2 are the extremities, and M the mid-point, of a chord in a continuous system of parallel chords of an ordinary curve, then the tangents to the curve at E_1 and E_2, and to the median at M, are concurrent.*

From this we gain immediately the

Corollary 1. *For a tangent chord like T_1S_1, the three tangents at T_1, S_1 and M are concurrent at S_1, so that the median touches the tangent chord at M. When T_1 is a point of inflexion, then M also is a point of inflexion of the median.*

Taking for τ a general direction (determined by slope m as before), and excluding multiple tangents, the preceding results may be summed up in

Theorem 2. *The median of an ordinary closed curve without multiple points associated with a given direction is an ordinary continuous curve without*

multiple points or cusps. Also the secondary medians are continuous and are always located between tangents that lie between A_τ and Z_τ. All medians are tangent to the corresponding tangent chords.

When T_1 is an inflexion, the secondary median in this neighborhood shrinks to the point T_1.

3. *Case of Double Tangent.* When there is a double tangent d_2 parallel to the given direction τ, such that d_2 besides the points of tangency has no other points in common with the closed curve, choose slightly different from τ a direction τ_a and construct in the neighborhood of d_2 the medians M_{τ_a} and S_{τ_a}, indicated by dash lines in Fig. 3. Now change the direction τ_a continuously

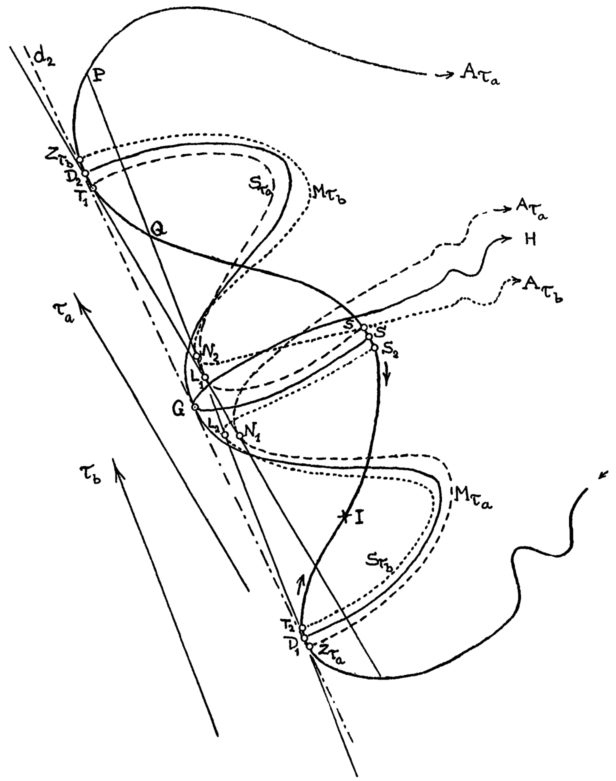

FIG. 3.

(in the figure clockwise) till it coincides with that of τ, and continue the change in the same sense to some direction τ_b also near that of τ. As τ_a approaches the direction of d_2, the points N_1 and L_1 both approach the mid-point G between the points of tangency D_1 and D_2 of d_2; and when τ_a coincides with the direction of d_2, then, according to Theorem 1, the curves M_τ and S_τ will both be tangent to d_2 at G. In the figure they are drawn as heavy lines: M_τ, from D_1 over G to H, results by continuous deformation from M_{τ_a}; S_τ, from D_2 over G to S, results continuously from S_{τ_a}. As τ_a continues to change over τ to τ_b, the characters of the curves M_{τ_a} and S_{τ_a} are partly interchanged. The part of M_{τ_a} from Z_{τ_a} to N_1 changes to the curve between D_1 and G, and from this to

the part $T_2 L_2$ of the curve S_{τ_b}. At the same time the portion $L_1 S_1$ of S_{τ_a} changes continuously into $L_2 S_2$ of S_{τ_b}. The remaining portions $N_1 A_{\tau_a}$ of M_{τ_a} and $L_1 T_1$ of S_{τ_a} change continuously into M_{τ_b}. Both M_{τ_b} and S_{τ_b} are indicated by dotted curves. From this it is seen that in the change from τ_a to τ_b *in the neighborhood of a double tangent as assumed, there is a continuous change in the system of medians M_τ and S_τ.*

In this change, M_{τ_a}, in passing over into M_{τ_b}, loses the portion $Z_{\tau_a} N_1$, while $N_1 A_{\tau_a}$ will remain. $Z_{\tau_a} N_1$ acquires $L_1 S_1$ to form after deformation S_{τ_b}. $N_1 A_{\tau_a}$, on the other hand, acquires $L_1 T_1$ to form M_{τ_b}.

4. *Continued Deformation of S_{τ_b}.* When τ continues to change from τ_b in the same sense as before, then the point of tangency T_2 of the corresponding tangent will move along the curve as indicated by the arrow, as long as there

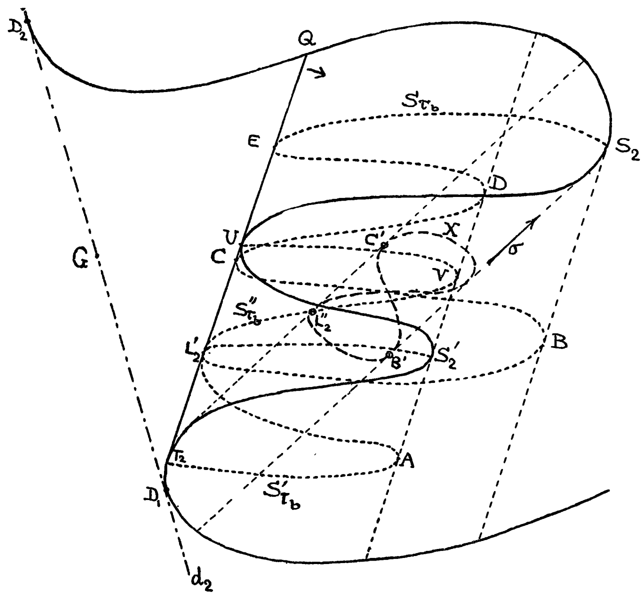

FIG. 4.

is no other intersection with the curve in the continuous change from $D_1 D_2$ to $T_2 Q$. In this case the extremities T_2 and S_2 of S_{τ_b} will move as indicated by the arrows. Assuming the uniqueness of Q in the continued deformation, S_2 and T_2 will simultaneously approach the point of inflexion I; and as the tangent, proceeding from T_2, approaches the tangency at I, S_{τ_b} shrinks to nothing at the inflexion I.

On the other hand, when, in the change of the tangent from T_2, other points of intersection than Q enter, this must occur the first time by a tangency, say at U (Fig. 4). The point of tangency L_2 of S_{τ_b} with $T_2 Q$ moves from G continuously to the position C in Fig. 4. The median S_{τ_b} consists of the arcs $T_2 A$, $A L_2'$, $L_2' B$, $B C$, $C D$, $D E$, $E S_2$. To the portion $T_2 L_2$ of S_{τ_b} in Fig. 3, corresponds the part of the curve, in Fig. 4, from T_2 over A, L_2' and B to C. Now, as $T_2 Q$ continues to move as a tangent to the closed curve in the neigh-

borhood of T_2 in the same sense, as indicated by the arrow, S_{τ_b} breaks up as in case of the double tangent d_2 in Fig. 3. After a slight deformation there will be a curve S'_{τ_b} arising from the arcs T_2A, AL'_2 of S_{τ_b} (Fig. 4), and a new arc $L'_2S'_2$. Another curve S''_{τ_b} arises from the remaining arcs L'_2B, BC, CD, DE, ES_2 of S_{τ_b} and the new arcs UV and VL'_2. In Fig. 3 we see that when the piece $Z_{\tau_a}N_1$ drops from M_{τ_a} in the deformation to M_{τ_b}, it is tangent to d_2 at D_1 and G. Likewise, at the moment when the piece T_1L_1 is added to M_{τ_b}, it is tangent to d_2 at D_2 and G. Similarly in Fig. 4. When $T_2AL'_2$ drops from S_{τ_b}, it is tangent to T_2Q at T_2 and L'_2. The remaining piece L'_2BC of the deformed part of M_{τ_a} that dropped in the position D_1G connected with the double tangent d_2, is also tangent to T_2Q at L'_2 and C. As to the further deformation of S'_{τ_b}, either the case of its shrinkage to the point of inflexion between T_2 and S'_2 may occur, or the same process as described in connection with Fig. 4 may repeat itself. In the continued deformation of S''_{τ_b} similar conditions arise. We need only pay attention to the piece L'_2BC. Portions may drop off or be added as explained above, or the whole section may remain as is the case in Fig. 4. As the direction of T_2Q moves to σ, L'_2BC is deformed into $L''_2B'C'$, where the points of tangency L''_2 and C' with the deformed tangent of T_2Q are preserved. Owing to a break-up in the deformation caused by the double tangent in the neighborhood of S_2 and S'_2, another arc joins $L''_2B'C'$, so as to form an 8-shaped curve X, which finally shrinks to a point. To sum up, we may state

THEOREM 3. *When a median M_τ breaks up on account of an external double tangent d_2, it loses a section that touches d_2 in two points. On the other hand, it gains a section that also touches d_2 in two points, of which one coincides with one of the points of tangency of the other section. A secondary median S_τ is either deformed in such a manner that it ultimately vanishes at a point of inflexion, or, owing to further double tangents, is broken up into sub-medians S'_τ, S''_τ, ..., such that the lost and remaining sections, like $T_2AL'_2$ and L'_2BC, always touch the corresponding tangent T_2Q in two points each.*

§ 3. *Case of Closed Curve Formed by Rectilinear Segments, generally by Analytic Arcs.*

A curve of this kind may contain reentrant angles, but shall have no double points, *i. e.*, points where two segments cross each other. Exactly as in the cases considered above, and in case of a convex polygon, it is found that with every direction τ is associated a median M_τ that extends from A_τ and Z_τ, such that the whole curve lies between the straight lines through A_τ and Z_τ parallel to τ. The extremities A_τ and Z_τ are either vertices of the polygon (curve) or mid-points of rectilinear segments. As in case of a convex polygon, a domain (P) exists entirely within the domain P enclosed by the curve, such

that among the entire system of medians M_τ, associated with all possible directions, no two medians intersect in the region within P and outside of (P).

In the construction of the median M_τ we must explain how to construct the segment of M_τ associated with a segment BC of a reentrant angle (Fig. 5) when BC is parallel to the direction τ, and when $ABCD$ and LK form parts of the contour. Draw $LL'' \parallel BC$, prolong BC to the intersection C'' (or B'') with KL, and let L' be the mid-point of LL'', B' that of BB''. Then, $B'L'$ is a portion of the median, and its prolongation passes through the intersection S of BA and KL produced. Also draw $KK'' \parallel BC$, and let K' be the mid-point of KK'' and C' that of CC''. Then, $B'C'$ and $C'K'$ are also portions of the median, and $B'C' = \frac{1}{2}BC$. Again, DC, $K'C'$ and KL produced are concurrent at a point T.

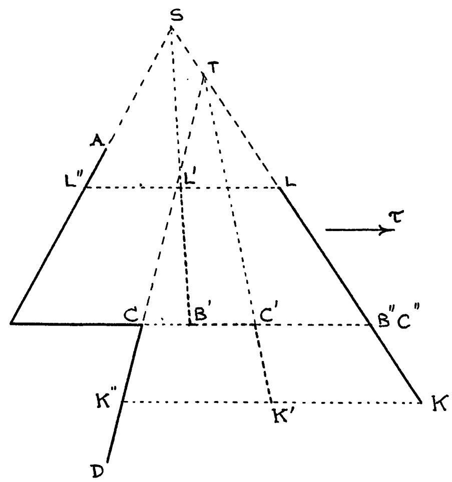

FIG. 5.

Next we must consider the variation of the median in the neighborhood of an external double tangent. The result is essentially the same as in case of Fig. 3, except that D_1 and D_2 are vertices of the polygon. D_1S and D_2S form now a reentrant angle; T_2 and Z_{τ_a} coincide with D_1; S_1 and S_2 with S; T_1 and Z_{τ_b} with D_2. We shall continue to call D_1, G, D_2 also in case of a polygon points of tangency. The variation of the median takes place according to the same principle, when one of the points D_1 or D_2 remains an ordinary tangency while the other is an angular point. Also the variation of secondary medians S_τ occurs according to the principles stated in connection with Fig. 5.

In Fig. 6 the change of M_{τ_a}, S_{τ_a} into M_{τ_b}, S_{τ_b} is shown for a special polygonal line AD_1SBCD, with the reentrant angle D_1SB and with the prolongation of CB or d_2 passing through D_1. Again, in the change from τ_a to τ_b, passing over d_2, M_{τ_a} loses the portion with its extremities at D_1 and G on d_2. (The notation and its meaning is the same as in Fig. 3.) On the other hand, the remainder of M_{τ_a} to pass into M_{τ_b} simply has to be joined to the piece

resulting from the portion of S_{τ_a} that, at the moment when τ coincided with the direction of d_2, had its extremities at D_2 and G in d_2.

When a closed curve without multiple points (where branches of the curve cross each other) formed by ordinary analytic arcs is given, we can inscribe to it a rectilinear polygon, as in case of a convex curve, and consider the given curve as the limit of such a polygon. As in the former case, the relations between M_σ and S_σ and their changes in the neighborhood of a line d_2 are not destroyed by the limit-process.

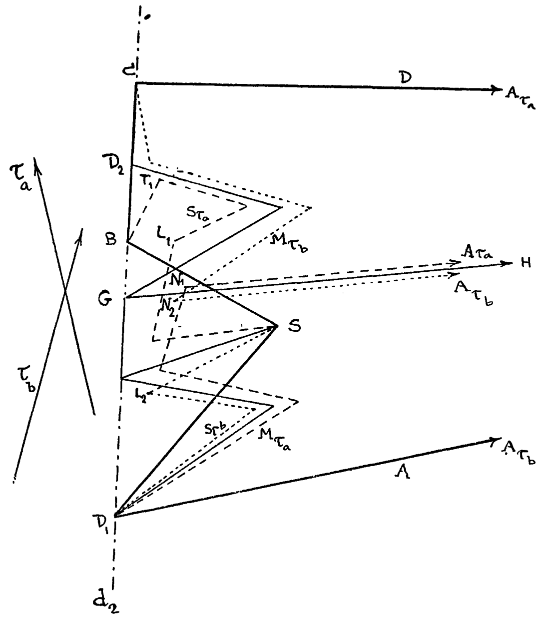

FIG. 6.

Calling a line d_2 that touches a curve in at least two distinct points, and with all points of the curve on one and the same side of d_2, a *base-line* of the curve, we can now state

THEOREM 4. *When τ changes continuously and in the same sense from τ_a to τ_b, two directions close to that of a base-line, and enclosing it, then, in the change of M_{τ_a} to M_{τ_b} associated with those directions and a closed curve formed by ordinary analytic arcs without multiple points, M_{τ_a} drops the portion that has its extremities in d_2, and becomes M_{τ_b} by joining to its remainder the portion of S_{τ_a} that has its extremities also in d_2.*

In the further deformation of the secondary median S_{τ_b}, due to a continuous change of τ from τ_b in the same direction, the same process repeats itself: the parts dropped off and those that are added have their extremities in the corresponding internal base-line ($D_1 Q$ in Fig. 4) resulting continuously from d_2.

§ 4. *Intersection of Medians Associated with a Pair of Orthogonal Directions.*

As has been shown in the previous papers, two medians M_σ and M_τ associated with two orthogonal directions σ and τ intersect in an odd number of points within the fundamental domain (R). It is evident that no extremity (A_σ, Z_σ) of one median can coincide with an extremity (A_τ, Z_τ) of the other, or lie in the other, and be the center of an inscribed rhomb. Hence, when, in a continuous change (Σ, T) of the pair (σ, τ) in which neither Σ nor T encloses a base-line, points of intersection of M_σ and M_τ disappear, they can not disappear separately. The only possibility is that two points first move to coincidence and then vanish. In other words, two distinct consecutive intersections of M_σ and M_τ must first coincide by M_σ and M_τ becoming tangent to each other. After the tangency M_σ and M_τ separate. Suppose next that σ approach the direction δ of an external base-line d_2. The points of M_τ in common with those of M_σ, when $\sigma = \delta$, if they exist, are the mid-points of chords, with the direction τ, that have one of their ends in the portion of the closed curve immediately above the base-line ($D_1 S D_2$ in Fig. 3).

Clearly within the domain (R)* M_τ can have no points in common with d_2. When $\sigma = \delta$, let M_a be the part which M_σ drops, and M_b the remainder. According to I, § 1, M_a and M_b cut M_τ respectively in an even and an odd number of points. Denoting by S_a the portion of S_σ with its extremities in d_2, then also S_a cuts M_τ in an even number of points. Hence, *in passing from the intersections of* M_{σ_a} *and* M_{τ_a} *to those of* M_{σ_b} *and* M_{τ_b} *in the neighborhood of a base-line, an even number of points are lost and an even number gained.* The same is true when a change of τ encloses the direction of a base-line, and when in a special case σ and τ simultaneously reach the directions of base-lines.

After σ continues to change from δ in the same sense, the even number of points of intersection of M_a and M_τ, which belongs to the original system, begins to decrease as M_a breaks up. The portion of M_τ that contains these points can have no points in common with the changed position of d_2, like $D_1 Q$ in Fig. 4. The points common to M_a and M_τ are the mid-points of chords having one of their extremities again on the portion of the closed curve directly supported by the interior base-line $D_1 Q$. The same argument applies to the cases where M_τ breaks up. Hence also the points of intersection of M_a and M_τ and possible further disintegrations, as long as they result from the original system of intersection by continuous change, when they exist, must disappear in pairs through previous tangencies of M_a and M_τ, or of parts resulting from them. We have therefore the

* When the curve has no other singular points than inflexions, this restriction is not necessary.

THEOREM 5. *In the change of the system of points of intersection of two medians associated with two continuously and uniformly changing orthogonal directions, points can disappear only after previous coincidence.*

As no point can disappear singly, when σ changes through a right angle, in general, these points in groups describe closed curves.

§ 5. *Classification of Points of Intersection of Medians, and Mode of Description of Closed Curves by these Points.*

A point of intersection of two medians M_σ and M_τ is either a center A_i of an inscribed rhomb of the first kind of the closed curve C, or a center B_k of an inscribed rhomb of the second kind. According to II, § 1, no point A_i can ever coincide with a point B_k through a tangency of M_σ and M_τ. The points A_i and the points B_k, if they exist simultaneously, form therefore two separate systems. Now the number of A_i's and B_k's is odd, so that the number of A_i's and the number of B_k's can not both be either even or odd; *i. e.*, when one is even, the other is odd.

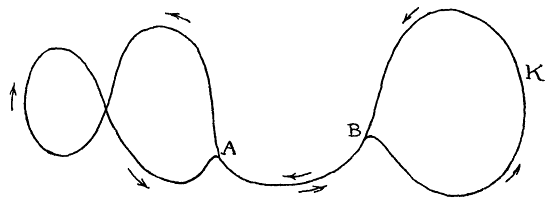

FIG. 7.

Suppose that the number of A_i's be odd. The A_i's in groups describe a finite number of closed curves, such that no points of one curve are permuted with points of any of the other curves. At least one of these groups has an odd number of points, which shall be denoted by $A_1, A_2, A_3, \ldots, A_n$ (n odd and $< i$). In general, every point of the closed curve K described by these points is the center of only one of the inscribed rhombs of the first kind. The curve K may have double points and double arcs as shown in Fig. 7. Every double point and point of a double arc is the center of two inscribed rhombs. Higher multiplicities yield accordingly corresponding numbers of rhombs with the same center.

Consider now a quarter of a full revolution of the direction σ. Obviously, since the couple (σ, τ) leads to the same set of points $A_1, A_2, \ldots, A_n$ as the couple (τ, σ), after a quarter-revolution of σ the same set of points appears. The points A are merely permuted according to a definite law that depends upon the shape of the curve. During a quarter-revolution the same two consecutive points A_λ and $A_{\lambda+1}$ can not vanish and reappear again through contact

of M_σ and M_τ without colliding with adjacent points. If they could, A_λ and $A_{\lambda+1}$ would then describe a closed curve which necessarily would be identical with the curve K; and A_λ and $A_{\lambda+1}$ would move into coincidence with other points of the set (A), since these describe parts of the same curve. Supposing that A_λ and $A_{\lambda+1}$ were describing only a part of K, and that T_1 were the point of contact of M_σ and M_τ where A_λ and $A_{\lambda+1}$ enter, and T_3 the point where they disappear through contact, then A_λ, to reach T_3 (Fig. 8), would have to pass over an infinite number of positions which $A_{\lambda+1}$ occupied before. There would therefore be an infinite number of points on K as centers of more than one inscribed rhomb. A_λ and $A_{\lambda+1}$ would describe a closed curve consisting of two coincident branches and forming a segment $\overrightarrow{XY}$ of the curve K. Denoting the remaining arc of K by $\leftarrow XY \rightarrow$, this part would have two free ends X and Y, which is not possible.

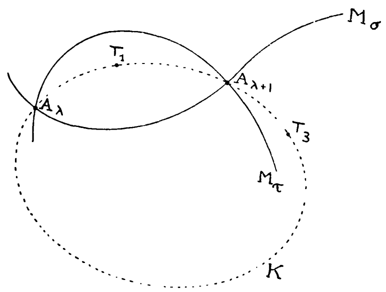

FIG. 8.

Any pair of orthogonal directions is contained in the set of all directions σ within a quarter-revolution and the corresponding orthogonal directions τ. Consequently, the set of points (A) describes the curve K just once when σ turns through a right angle.

In the continuous change from σ to τ an even number of points (E) of the set (A) vanish and reappear through contact of M_σ and M_τ. Consequently, in the same set (A) an odd number of points (F) exists, such that each of these points moves to a neighboring point of the set (A) during the change from σ to τ. Exactly the same results are obtained with respect to a group of an odd number of B_k's generating a closed curve.*

* In the deformation of secondary medians that result from the breaking up an M_σ, parts S' may enter whose continuous deformations never will form a part of an M_σ. In the deformation of S', its free end on K may meet M_T in a point L. In the further deformation the intersection of S' and M_T, if it exists, describes a curve with two free ends L and L_1 on C. The points of such a curve are also centers of inscribed rhombs, but there is no continuous connection between these points and the points of a curve K. The points L and L_1 are centers of degenerate rhombs in which a pair of opposite vertices coincides with L in one case and with L_1 in the other. Points of such a curve may even be centers of inscribed squares, although this is not in general true. As cases of this kind do not affect the set (A) and the curves described by its points, no further attention will be given them.

§ 6. *System of Inscribed Rhombs Associated with a Curve K.*

Consider again the curve K described by the set (A). Denote the rhomb associated with the center A_i by R_1^i, R_2^i, R_3^i, R_4^i. Let A_λ, $A_{\lambda+1}$ be two points of the type (E) associated with the initial direction σ. To the change from σ to τ corresponds a set of rhombs connecting continuously $R_1^\lambda R_2^\lambda R_3^\lambda R_4^\lambda$ and $R_1^{\lambda+1} R_2^{\lambda+1} R_3^{\lambda+1} R_4^{\lambda+1}$ in the given order. We say that in this case the two rhombs are directly connected, and that the continuous set connecting them is a *direct set*. Taking two points A_μ, $A_{\mu+1}$ of the type (F), then, in the change from σ to τ, as A_μ moves to $A_{\mu+1}$, $R_1^\mu R_2^\mu R_3^\mu R_4^\mu$ changes continuously into $R_1^{\mu+1} R_2^{\mu+1} R_3^{\mu+1} R_4^{\mu+1}$, such that after σ has reached τ, R_1^μ coincides with $R_2^{\mu+1}$, R_2^μ with $R_3^{\mu+1}$, R_3^μ with $R_4^{\mu+1}$, R_4^μ with $R_1^{\mu+1}$ (or, as another possibility: R_1^μ with $R_4^{\mu+1}$, R_2^μ with $R_1^{\mu+1}$, R_3^μ with $R_2^{\mu+1}$, R_4^μ with $R_3^{\mu+1}$). The two rhombs are connected by a continuous set of rhombs that permutes the original order of the diagonals. The two rhombs are said to be indirectly connected, and the set connecting them is an *indirect set*. Now, according to § 5, the number of direct sets is even, that of indirect sets odd. The effect of two indirect sets is evidently the same as that of a direct set. The effect of all direct and indirect sets is therefore that of an indirect set. Consequently, if we start from the rhomb $R_1 R_2 R_3 R_4$ with the center A_1, we can in succession pass through a continuous set of rhombs to those with centers $A_2, A_3, \ldots, A_n$, and finally back to the original rhomb in the order $R_2 R_3 R_4 R_1$ (or $R_4 R_1 R_2 R_3$). In other words, $R_1 R_2 R_3 R_4$ can be continuously deformed into itself such that $R_1 R_2 R_3 R_4$ will respectively move to $R_2 R_3 R_4 R_1$.

The same is true with respect to a set (B) of the second kind and of the same type.

We can therefore conclude as before, that *among such a system of rhombs there must be at least one square.*

But it is established that at least one such system, either of the first or of the second kind, always exists. There may, of course, be several systems; for example, one group of an odd number of A's and two groups each of an odd number of B's. In this case there would be at least one inscribed square of the first and two inscribed squares of the second kind.

The theorem on the inscribed square of a closed curve is therefore proved for any closed curve formed by a finite number of analytic arcs without double points and with a finite number of inflexions and other singularities.

The restriction on double points can, however, immediately be removed. From a closed curve with multiple points we can always in a number of ways detach closed curves without multiple points, to which, according to the foregoing result, a square may be inscribed. Hence the

THEOREM 6. *It is always possible to inscribe at least one square in a closed curve formed by a finite number of analytic arcs with a finite number of inflexions and other singularities.*